REFLECTION EQUATION, TWIST, AND EQUIVARIANT QUANTIZATION

BY

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ABSTRACT

We prove that the reflection equation (RE) algebra $\mathcal{L}_R$ associated with a finite dimensional representation of a quasitriangular Hopf algebra $\mathcal{H}$ is twist-equivalent to the corresponding Faddeev–Reshetikhin–Takhtajan (FRT) algebra. We show that $\mathcal{L}_R$ is a module algebra over the twisted tensor square $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$ and the double D($\mathcal{H}$). We define FRT- and RE-type algebras and apply them to the problem of equivariant quantization on Lie groups and matrix spaces.

1. Introduction

Let $\mathcal{H}$ be a quasitriangular Hopf algebra with the universal R-matrix $\mathcal{R}$. Let V be the space of its finite dimensional representation and R the image of $\mathcal{R}$ in $\mathrm{End}^{\odot 2}(V)$. We study relations between two algebras naturally arising in this context, the Faddeev–Reshetikhin–Takhtajan (FRT) algebra $\mathcal{T}_R$ and the so-called reflection equation (RE) algebra $\mathcal{L}_R$. They are both quotients of the tensor algebra $\mathrm{T}(\mathrm{End}^*(V))$ by quadratic relations and admit certain Hopf algebra actions. So, $\mathcal{T}_R$ is endowed with the structure of a bimodule algebra over $\mathcal{H}$ that is naturally extended from the bimodule structure on $\mathrm{End}^*(V)$. The algebra $\mathcal{L}_R$ is a left $\mathcal{H}$-module algebra with the action extended from the coadjoint representation on $\mathrm{End}^*(V)$.

Our first result is that the RE algebra $\mathcal{L}_R$ is a module algebra not only over $\mathcal{H}$ but over the twisted tensor square $\mathcal{H}\overset{R}{\otimes}\mathcal{H}$. The latter is the twist of the ordinary tensor product $\mathcal{H} \otimes \mathcal{H}$ with the universal R-matrix as a twisting cocycle, $\mathcal{R}_{23} \in (\mathcal{H} \otimes \mathcal{H}) \otimes (\mathcal{H} \otimes \mathcal{H})$. The action of $\mathcal{H}$ on $\mathcal{L}_R$ is induced by the Hopf algebra embedding $\mathcal{H} \to \mathcal{H}\overset{R}{\otimes}\mathcal{H}$ via the coproduct.

As a corollary, we obtain that $\mathcal{L}_R$ is a module algebra over the coopposite dual $\mathcal{H}^{*op}$ because there exists a Hopf homomorphism $\mathcal{H}^{*op} \to \mathcal{H}\overset{R}{\otimes}\mathcal{H}$. Since the Hopf algebra homomorphisms from $\mathcal{H}$ and $\mathcal{H}^{*op}$ to $\mathcal{H}\overset{R}{\otimes}\mathcal{H}$ can be extended to a Hopf algebra homomorphism from the double $D(\mathcal{H})$ to $\mathcal{H}\overset{R}{\otimes}\mathcal{H}$, we obtain that $\mathcal{L}_R$ is a module algebra over $D(\mathcal{H})$.

Our second result is that $\mathcal{L}_R$ is a twist of $\mathcal{T}_R$ as a module algebra. The $\mathcal{H}$-bimodule $\mathcal{T}_R$ can be considered as a left $\mathcal{H}^{op} \otimes \mathcal{H}$-module. The twist from $\mathcal{H}^{op} \otimes \mathcal{H}$ to $\mathcal{H}\overset{R}{\otimes}\mathcal{H}$ is performed with the help of the cocycle $\mathcal{R}_{13}\mathcal{R}_{23}$, where the first transformation via $\mathcal{R}_{13}$ converts the first tensor factor $\mathcal{H}^{op}$ to $\mathcal{H}$ while the second twist via $\mathcal{R}_{23}$ makes the ordinary tensor square $\mathcal{H}^{\otimes 2}$ the twisted one.

The algebra $\mathcal{T}_R$ is commutative in the category of $\mathcal{H}$-bimodules. Using this fact, we prove that $\mathcal{L}_R$ is commutative in the category of $\mathcal{H}\overset{R}{\otimes}\mathcal{H}$-modules. In general, we prove twist-equivalence between the classes of FRT- and RE-type algebras, which we define to be commutative algebras in the categories of $\mathcal{H}$-bimodules and $\mathcal{H}\overset{R}{\otimes}\mathcal{H}$-modules, respectively.

In particular, we introduce the RE dual algebra $\tilde{\mathcal{H}}^*$ as an RE-type algebra that is twist-equivalent to the FRT-type algebra $\mathcal{H}^*$ and we show that it coincides with the braided Hopf algebra of Majid.

We study coactions on $\tilde{\mathcal{H}}^*$ of the Hopf algebras $(\mathcal{H}\overset{R}{\otimes}\mathcal{H})^*$, $\mathcal{H}^*$, and the opposite Hopf algebra $\mathcal{H}_{op}$. We deduce properties of the $\mathcal{H}$-equivariant homomorphism $\tilde{\mathcal{H}}^* \to \mathcal{H}$, $\xi \mapsto \langle\xi, \mathcal{Q}_1\rangle\mathcal{Q}_2$, where $\mathcal{Q} = \mathcal{R}_{21}\mathcal{R}$, using the coalgebra structure over $\mathcal{H}_{op}$.

We apply our construction to the deformation quantization on Lie groups and matrix spaces. In particular, we show that the algebra $\tilde{\mathcal{U}}_h^*(\mathfrak{g})$, where $\mathfrak{g}$ is a semisimple Lie algebra, is the $\mathcal{U}_h(\mathfrak{g})$-equivariant quantization of a special Poisson structure on the corresponding Lie group, the RE bracket. It is known that the quotient of $\mathcal{T}_R$ by torsion is the quantization on the cone $\mathrm{End}^\Omega(V)$ of matrices whose tensor square commutes with the image Ω of the split Casimir, the invariant symmetric element from $\mathfrak{g}^{\otimes 2}$. As an implication of the twist-equivalence between $\mathcal{L}_R$ and $\mathcal{T}_R$, we find that the quotient of $\mathcal{L}_R$ by torsion is the $\mathcal{U}_h(\mathfrak{g})\overset{R}{\otimes}\mathcal{U}_h(\mathfrak{g})$-equivariant quantization on $\mathrm{End}^\Omega(V)$.

The setup of the paper is as follows. Section 2 contains basic facts about quasi-

triangular Hopf algebras, the twist transformation, and the relation between the double and twisted tensor square. Section 3 recalls what are modules and co-modules over Hopf algebras. The relations between the FRT and RE algebra and their implications are studied in Section 4. Section 5 is devoted to applications to the equivariant deformation quantization on Lie groups and matrix spaces.

2. Quasitriangular Hopf algebras

2.1. DEFINITIONS AND ELEMENTARY PROPERTIES. In this subsection, we recall basic definitions of the quasitriangular Hopf algebra theory, [Dr1]. For a detailed exposition, the reader can consult [Mj]. Let $\mathbb{K}$ be a commutative algebra over a field of zero characteristic. Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{K}$, with the coproduct $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, counit $\varepsilon\colon \mathcal{H} \to \mathbb{K}$, and antipode $\gamma\colon \mathcal{H} \to \mathcal{H}$. Throughout the paper, we adopt the standard notation with implicit summation in order to explicate factors of tensor objects, e.g., we write $\Phi = \Phi_1 \otimes \cdots \otimes \Phi_k$ for an element $\Phi \in \mathcal{H}^{\otimes k}$. For the coproduct, we use the symbolic Sweedler notation, $\Delta(x) = x_{(1)} \otimes x_{(2)}$, $x \in \mathcal{H}$.

A Hopf algebra $\mathcal{H}$ is called **quasitriangular** if there is an element $\mathcal{R} \in \mathcal{H}^{\otimes 2}$, the universal R-matrix, such that

$$(1) \qquad (\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$

and, for any $x \in \mathcal{H}$,

$$(2) \qquad \mathcal{R}\Delta(x) = \Delta^{op}(x)\mathcal{R}.$$

The subscripts in (1) specify the way of embedding $\mathcal{H}^{\otimes 2}$ into $\mathcal{H}^{\otimes 3}$, namely, $\mathcal{R}_{12} = \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes 1$, $\mathcal{R}_{13} = \mathcal{R}_1 \otimes 1 \otimes \mathcal{R}_2$, and $\mathcal{R}_{23} = 1 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2$. The opposite coproduct Δ^{op} is the composition of Δ and the flip operator τ on $\mathcal{H}^{\otimes 2}$.

The following identities are implications of defining relations (1) and (2):

$$(3) \qquad (\varepsilon \otimes id)(\mathcal{R}) = 1 \otimes 1, \quad (id \otimes \varepsilon)(\mathcal{R}) = 1 \otimes 1,$$
$$(4) \qquad (\gamma \otimes id)(\mathcal{R}) = \mathcal{R}^{-1}, \quad (id \otimes \gamma)(\mathcal{R}^{-1}) = \mathcal{R}.$$

Also, the Yang–Baxter equation in $\mathcal{H}^{\otimes 3}$,

$$(5) \qquad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},$$

follows from (1) and (2).

There is an alternative quasitriangular structure[1] on $\mathcal{H}$ with the universal R-matrix $\mathcal{R}_{21}^{-1} = \tau(\mathcal{R}^{-1})$. Obviously, it fulfills conditions (1) and (2).

The opposite Hopf algebra $\mathcal{H}_{op}$ is endowed with the opposite multiplication $m_{op} = m \circ \tau$, where m is the original one. Similarly, the coopposite Hopf algebra $\mathcal{H}^{op}$ has the coproduct $\Delta^{op} = \tau \circ \Delta$. The antipode of $\mathcal{H}$ is an isomorphism between $\mathcal{H}$ and $\mathcal{H}_{op}^{op}$, being an anti-algebra and anti-coalgebra map. Important for our exposition is that we may also treat it as an isomorphism between $\mathcal{H}_{op}$ and $\mathcal{H}^{op}$.

The dual Hopf algebra $\mathcal{H}^*$ is spanned by matrix coefficients of all finite dimensional representations[2] of $\mathcal{H}$. There are two remarkable maps from $\mathcal{H}^{*op}$ to $\mathcal{H}$ defined via the universal R-matrix:

(6) $\mathcal{R}^{\pm}(\eta) = \langle \eta, \mathcal{R}_1^{\pm} \rangle \mathcal{R}_2^{\pm}, \quad \eta \in \mathcal{H}^{*op}, \quad \text{where} \quad \mathcal{R}^+ = \mathcal{R} \quad \text{and} \quad \mathcal{R}^- = \mathcal{R}_{21}^{-1}.$

It follows from identity (1) that they are Hopf algebra homomorphisms $\mathcal{H}^{*op} \to \mathcal{H}$ (as was already mentioned, the element $\mathcal{R}_{21}^{-1}$ satisfies (1) as well).

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of two Hopf algebras $\mathcal{A}$ and $\mathcal{B}$ is a Hopf algebra with the multiplication

(7) $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2, \quad a_i \in \mathcal{A}, \ b_i \in \mathcal{B}, \ i = 1, 2,$

coproduct

(8) $\Delta(a \otimes b) = (a_{(1)} \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)}), \quad a \in \mathcal{A}, \ b \in \mathcal{B},$

counit $\varepsilon = \varepsilon_{\mathcal{A}} \otimes \varepsilon_{\mathcal{B}}$, and antipode $\gamma = \gamma_{\mathcal{A}} \otimes \gamma_{\mathcal{B}}$. If $\mathcal{A}$ and $\mathcal{B}$ are both quasitriangular, so is $\mathcal{A} \otimes \mathcal{B}$. Its universal R-matrix is

(9) $\mathcal{R}_{\mathcal{A} \otimes \mathcal{B}} = \mathcal{R}_{\mathcal{A}} \mathcal{R}_{\mathcal{B}},$

where $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{B}}$ are the R-matrices of $\mathcal{A}$ and $\mathcal{B}$ naturally embedded in $(\mathcal{A} \otimes \mathcal{B})^{\otimes 2}$.

Remark 2.1: In the infinite dimensional case, we assume that all algebras are complete in some topology. It may be, for example, the h-adic topology in the case $\mathbb{K} = \mathbb{C}[[h]]$. All tensor products are assumed to be completed.

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1 They may coincide.

2 We assume that the supply of representations is large enough to separate elements of $\mathcal{H}$.

2.2. TWIST OF HOPF ALGEBRAS. In this subsection, we collect several facts and examples, which will be essential for our further exposition, concerning the twist transformation of Hopf algebras, [Dr2]. Let $\mathcal{F}$ be an invertible element from $\mathcal{H} \odot \mathcal{H}$ satisfying the cocycle constraint

$$(10) \qquad (\Delta \odot \mathrm{id})(\mathcal{F})\mathcal{F}_{12} = (\mathrm{id} \odot \Delta)(\mathcal{F})\mathcal{F}_{23},$$

with the normalizing condition $(\varepsilon \otimes \mathrm{id})(\mathcal{F}) = 1 \odot 1 = (\mathrm{id} \otimes \varepsilon)(\mathcal{F})$. There exists a new Hopf algebra structure $\tilde{\mathcal{H}}$ on $\mathcal{H}$ with the same multiplication and counit but the "twisted" coproduct

$$(11) \qquad \tilde{\Delta}(x) = \mathcal{F}^{-1}\Delta(x)\mathcal{F}, \quad x \in \mathcal{H},$$

and antipode

$$\tilde{\gamma}(x) = u^{-1}\gamma(x)u, \quad \text{where } u = \gamma(\mathcal{F}_1)\mathcal{F}_2 \in \mathcal{H}.$$

Condition (10) ensures the coproduct $\tilde{\Delta}$ being coassociative. The Hopf algebra $\tilde{\mathcal{H}}$ is quasitriangular, provided so is $\mathcal{H}$. Its universal R-matrix is expressed through the old one and the twisting cocycle:

$$(12) \qquad \tilde{\mathcal{R}} = \mathcal{F}_{21}^{-1}\mathcal{R}\mathcal{F}.$$

We call algebras $\mathcal{H}$ and $\tilde{\mathcal{H}}$ **twist-equivalent** and use the notation $\tilde{\mathcal{H}} \overset{\mathcal{F}}{\sim} \mathcal{H}$ or simply $\tilde{\mathcal{H}} \sim \mathcal{H}$ when the exact form of $\mathcal{F}$ is clear from the context. Obviously, $\mathcal{H}_2 \overset{\mathcal{F}_1}{\sim} \mathcal{H}_1$ and $\mathcal{H}_3 \overset{\mathcal{F}_2}{\sim} \mathcal{H}_2$ imply $\mathcal{H}_3 \overset{\mathcal{F}_1\mathcal{F}_2}{\sim} \mathcal{H}_1$. Also, if $\mathcal{H}_2 \overset{\mathcal{F}}{\sim} \mathcal{H}_1$, then $\mathcal{H}_1 \overset{\mathcal{F}^{-1}}{\sim} \mathcal{H}_2$. Essential for us will be the following examples.

Example 2.2 (Coopposite Hopf algebra): Given a quasitriangular Hopf algebra $\mathcal{H}$ its coopposite algebra $\mathcal{H}^{op}$ can be obtained by twist with $\mathcal{F} = \mathcal{R}^{-1}$, cf. equation (2). The cocycle condition (10) follows from the Yang–Baxter equation (5).

Example 2.3 (Twisted tensor product): Let $\mathcal{H}$ be the tensor product $\mathcal{H} = \mathcal{A} \odot \mathcal{B}$ of two Hopf algebras with multiplication (7) and coproduct (8). An element $\mathcal{F} \in \mathcal{B} \odot \mathcal{A}$ may be viewed as that from $\mathcal{H} \odot \mathcal{H}$ via the embedding $(1 \otimes \mathcal{B}) \odot (\mathcal{A} \odot 1) \subset (\mathcal{A} \odot \mathcal{B}) \odot (\mathcal{A} \odot \mathcal{B})$. If $\mathcal{F}$ satisfies the identities

$$(13) \quad (\Delta_{\mathcal{B}} \odot \mathrm{id})(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23} \in \mathcal{B} \odot \mathcal{B} \odot \mathcal{A}, \quad (\mathrm{id} \odot \Delta_{\mathcal{A}})(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{12} \in \mathcal{B} \odot \mathcal{A} \odot \mathcal{A},$$

it also fulfills the cocycle condition (10), [RS].

Definition 2.4: **Twisted tensor product** $\mathcal{A}\overset{\mathcal{F}}{\otimes}\mathcal{B}$ of two Hopf algebras is the twist of $\mathcal{A}\otimes\mathcal{B}$ with a cocycle $\mathcal{F}$ satisfying (13).

An immediate corollary of conditions (13) is that the evaluation maps $\mathrm{id}\otimes\varepsilon_{\mathcal{B}}$: $\mathcal{A}\overset{\mathcal{F}}{\otimes}\mathcal{B}\to\mathcal{A}$ and $\varepsilon_{\mathcal{A}}\otimes\mathrm{id}$: $\mathcal{A}\overset{\mathcal{F}}{\otimes}\mathcal{B}\to\mathcal{B}$ are Hopf ones. Note that, contrary to the ordinary tensor product, the embeddings of $\mathcal{A}$ and $\mathcal{B}$ into $\mathcal{A}\overset{\mathcal{F}}{\otimes}\mathcal{B}$ are algebra but not coalgebra maps.

An important example of the twisted tensor product is when $\mathcal{A}=\mathcal{B}=\mathcal{H}$ is a quasitriangular Hopf algebra and $\mathcal{F}=\mathcal{R}$. Condition (13) then holds because of (1). This particular case is called **twisted tensor square** of $\mathcal{H}$ and denoted $\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$. It is convenient for our exposition to take the universal R-matrix $\mathcal{R}_{13}^{-}\mathcal{R}_{24}^{+}$ for the ordinary tensor product $\mathcal{H}\odot\mathcal{H}$, see (6) and (9). Then formula (12) gives the universal R-matrix of $\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$:

$$(14)\qquad \mathcal{R}_{41}^{-1}\mathcal{R}_{13}^{-}\mathcal{R}_{24}^{+}\mathcal{R}_{23}=\mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}\in(\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H})\otimes(\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}).$$

2.3. DRINFELD'S DOUBLE AND TWISTED TENSOR SQUARE. This subsection is devoted to a relation between the double $\mathrm{D}(\mathcal{H})$ of a quasitriangular Hopf algebra $\mathcal{H}$ and its twisted tensor square $\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$. A particular case of the so-called factorizable Hopf algebras was considered in [RS]. For those algebras, the double $\mathrm{D}(\mathcal{H})$ is isomorphic to $\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$. In general, there is a Hopf algebra homomorphism from $\mathrm{D}(\mathcal{H})$ to $\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$.

As a coalgebra, the double $\mathrm{D}(\mathcal{H})$ coincides with the tensor product $\mathcal{H}\odot\mathcal{H}^{*op}$. Both of $\mathcal{H}$ and $\mathcal{H}^{*op}$ are embedded in $\mathrm{D}(\mathcal{H})$ as Hopf subalgebras. The cross-commutation relations between elements from the two tensor factors can be written in the form

$$(15)\qquad \langle\eta_{(1)},x_{(1)}\rangle\eta_{(2)}x_{(2)}=\langle\eta_{(2)},x_{(2)}\rangle x_{(1)}\eta_{(1)},$$

for any $x\in\mathcal{H}$, $\eta\in\mathcal{H}^{*op}$. In the finite dimensional case, they are equivalent to the Yang–Baxter equation on the canonical element $\sum_{i}e_{i}\otimes e^{i}$, where $\{e_{i}\}\subset\mathcal{H}$ and $\{e^{i}\}\subset\mathcal{H}^{*op}$ are dual bases. Then $\mathrm{D}(\mathcal{H})$ is dual to the twisted tensor product $\mathcal{H}^{*}\overset{\mathcal{F}}{\otimes}\mathcal{H}_{op}$, where $\mathcal{F}=\sum_{i}e_{i}\otimes e^{i}$ is considered as an element from $\mathcal{H}_{op}\otimes\mathcal{H}^{*}$.

PROPOSITION 2.5: *Let $\mathcal{H}$ be a quasitriangular Hopf algebra. The coproduct*

$$(16)\qquad\qquad \mathcal{H}\xrightarrow{\Delta}\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$$

and the composite map

$$(17)\qquad\qquad \mathcal{H}^{*op}\xrightarrow{\Delta}\mathcal{H}^{*op}\otimes\mathcal{H}^{*op}\xrightarrow{\mathcal{R}^{+}\otimes\mathcal{R}^{-}}\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$$

are *Hopf algebra homomorphisms.*

Proof: Straightforward. ∎

Homomorphisms (16) and (17) may be extended to $D(\mathcal{H})$. For finite dimensional Hopf algebras, a proof of this statement can be found in [Mj]. Infinite dimensional Hopf algebras like quantum groups are of primary interest for this article, and we present here a proof suitable for the general case.

THEOREM 2.6: *Let $\mathcal{H}$ be a quasitriangular Hopf algebra. Then, maps (16) and (17) define a Hopf homomorphism $D(\mathcal{H}) \to \mathcal{H} \overset{\mathcal{R}}{\odot} \mathcal{H}$.*

Proof: As a linear space, the double coincides with the tensor product of $\mathcal{H}$ and $\mathcal{H}^{*op}$, which are embedded in $D(\mathcal{H})$ as Hopf subalgebras. Taking Proposition 2.5 into account, it suffices to show that the permutation relations between elements of $\mathcal{H}$ and $\mathcal{H}^{*op}$ are respected. Applying maps (16) and (17) to both sides of identity (15), we come to the equation

$$\langle \eta, \mathcal{R}_1^- \mathcal{R}_1^+ x_{(1)} \rangle \mathcal{R}_2^+ x_{(2)} \odot \mathcal{R}_2^- x_{(3)} = \langle \eta, x_{(3)} \mathcal{R}_1^- \mathcal{R}_1^+ \rangle x_{(1)} \mathcal{R}_2^+ \odot x_{(2)} \mathcal{R}_2^-$$

that must hold for any $x \in \mathcal{H}$ and $\eta \in \mathcal{H}^{*op}$. It is fulfilled indeed, because both of the elements $\mathcal{R}^{\pm}$ satisfy equation (2). ∎

3. Modules over Hopf algebras

3.1. MODULE ALGEBRAS.   This subsection contains some facts about the modules over a Hopf algebra $\mathcal{H}$. An associative algebra $\mathcal{A}$ is called a left $\mathcal{H}$-**module algebra** if the multiplication $\mathcal{A} \odot \mathcal{A} \to \mathcal{A}$ is a homomorphism of $\mathcal{H}$-modules. Similarly, one can consider right modules over $\mathcal{H}$. Explicitly, for any $x \in \mathcal{H}$ and $a, b \in \mathcal{A}$ the consistency conditions read

(18) $x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b), (ab) \triangleleft x = (a \triangleleft x_{(1)})(b \triangleleft x_{(2)}),$

(19) $1 \triangleright a = a,\ x \triangleright 1_A = \varepsilon(x) 1_A, a \triangleleft 1 = a,\ 1_A \triangleleft x = \varepsilon(x) 1_A,$

for the left and right actions $\triangleright$ and $\triangleleft$. If $\mathcal{A}$ is simultaneously a left and right module and the two actions commute,

(20) $x_1 \triangleright (a \triangleleft x_2) = (x_1 \triangleright a) \triangleleft x_2, \quad x_1, x_2 \in \mathcal{H}, \quad a \in \mathcal{A},$

then it is called **bimodule**. $\mathcal{A}$ is an $\mathcal{H}$-**bimodule algebra** if its bimodule and algebra structures are consistent in the sense of (18)–(19).

Example 3.1 (Adjoint action): A Hopf algebra $\mathcal{H}$ is a left and right module algebra over itself with respect to the left and right adjoint actions

$$(21) \qquad \mathrm{ad}(x) \triangleright y = x_{(1)} y \gamma(x_{(2)}), \quad y \triangleleft \mathrm{ad}(x) = \gamma(x_{(1)}) y x_{(2)},$$

for $x, y \in \mathcal{H}$.

Example 3.2 (Dual Hopf algebra $\mathcal{H}^*$): A Hopf algebra $\mathcal{H}$ is a bimodule over itself with respect to the regular actions by multiplication from the left and right. However, these actions do not respect the multiplication in $\mathcal{H}$. On the contrary, the dual (coregular) actions are consistent with the multiplication in $\mathcal{H}^*$, so the latter is an $\mathcal{H}$-bimodule algebra. Explicitly, the coregular actions can be expressed via the coproduct in $\mathcal{H}^*$ and the pairing $\langle .,. \rangle$ between $\mathcal{H}^*$ and $\mathcal{H}$:

$$(22) \qquad x \triangleright a = a_{(1)} \langle a_{(2)}, x \rangle, \quad a \triangleleft x = \langle a_{(1)}, x \rangle a_{(2)},$$

where $x \in \mathcal{H}$ and $a \in \mathcal{H}^*$.

Example 3.3: Let $\mathcal{H}$ be a Hopf algebra and $\mathcal{A}$ its bimodule algebra. Then $\mathcal{A}$ is a left $\mathcal{H}_{op} \odot \mathcal{H}$-module algebra with the action

$$(23) \qquad (x \otimes y) \triangleright a = y \triangleright a \triangleleft x, \quad x \odot y \in \mathcal{H}_{op} \odot \mathcal{H}, \ a \in \mathcal{A}.$$

It is a left $\mathcal{H}^{op} \otimes \mathcal{H}$-module algebra with the action

$$(24) \qquad (x \odot y) \triangleright a = y \triangleright a \triangleleft \gamma(x), \quad x, y \in \mathcal{H}^{op} \odot \mathcal{H}, \ a \in \mathcal{A}.$$

This example means that we may consider only left modules, instead of bimodules.

Twist of Hopf algebras induces a transformation of their module algebras. Let $\mathcal{A}$ be an $\mathcal{H}$-module algebra with the multiplication m and let $\tilde{\mathcal{H}} \overset{\mathcal{F}}{\sim} \mathcal{H}$. The new associative multiplication

$$(25) \qquad \tilde{m}(a \odot b) = m(\mathcal{F}_1 \triangleright a \otimes \mathcal{F}_2 \triangleright b), \quad a, b \in \mathcal{A},$$

can be introduced on $\mathcal{A}$. We denote this algebra by $\tilde{\mathcal{A}}$. Since $\tilde{\mathcal{H}} \simeq \mathcal{H}$ as associative algebras, the action of $\mathcal{H}$ on $\mathcal{A}$ can be viewed as that of $\tilde{\mathcal{H}}$ on $\tilde{\mathcal{A}}$. This action is consistent with multiplication (25) in $\tilde{\mathcal{A}}$ and twisted coproduct (11) in $\tilde{\mathcal{H}}$. We say that $\tilde{\mathcal{A}}$ and $\mathcal{A}$ are **twist-equivalent** and write $\tilde{\mathcal{A}} \overset{\mathcal{F}}{\sim} \mathcal{A}$, by analogy with the Hopf algebras. Also, we shall omit $\mathcal{F}$ and write simply $\tilde{\mathcal{A}} \sim \mathcal{A}$ if the exact form of $\mathcal{F}$ is clear from the context.

3.2. COMODULE ALGEBRAS.   Let $\mathcal{H}$ be a Hopf algebra and $\mathcal{H}^*$ its dual. A right $\mathcal{H}^*$-comodule algebra is an associative algebra $\mathcal{A}$ endowed with a homomorphism $\delta\colon \mathcal{A} \to \mathcal{A} \odot \mathcal{H}^*$ obeying the coassociativity constraint

$$(26) \qquad\qquad (\mathrm{id} \odot \Delta) \circ \delta = (\delta \odot \mathrm{id}) \circ \delta$$

and the conditions

$$(27) \qquad\qquad \delta(1_{\mathcal{A}}) = 1_{\mathcal{A}} \odot 1, \quad (\mathrm{id} \otimes \varepsilon) \circ \delta = \mathrm{id},$$

where the identity map on the right-hand side assumes the isomorphism $\mathcal{A} \otimes \mathbb{K} \simeq \mathcal{A}$. As for the coproduct Δ, we use symbolic notation $\delta(a) = a_{[1]} \odot a_{(2)}$, marking the tensor component belonging to $\mathcal{A}$ with the square brackets; the subscript of the $\mathcal{H}^*$-component is concluded in parentheses. Every right $\mathcal{H}^*$-comodule $\mathcal{A}$ is a left $\mathcal{H}$-module, the action being defined through the pairing $\langle .,. \rangle$ between $\mathcal{H}$ and $\mathcal{H}^*$:

$$(28) \qquad\qquad x \triangleright a = a_{[1]} \langle a_{(2)}, x \rangle, \quad x \in \mathcal{H},\, a \in \mathcal{A}.$$

Suppose there is a map δ from an $\mathcal{H}$-module algebra $\mathcal{A}$ to $\mathcal{A} \odot \mathcal{H}^*$ such that for any $x \in \mathcal{H}$, $a \in \mathcal{A}$, and a linear functional $\alpha \in \mathcal{A}^*$

$$(29) \qquad \alpha(x \triangleright a) = \alpha(a_{[1]}) \langle a_{(2)}, x \rangle, \quad \text{where } \delta(a) = a_{[1]} \odot a_{(2)}.$$

Then $\mathcal{A}$ is an $\mathcal{H}^*$-comodule algebra with the coaction δ. Note that if $\mathcal{H}$ or $\mathcal{A}$ are finite dimensional, equation (29) may serve as a definition of δ, so the notions of $\mathcal{H}$-module and $\mathcal{H}^*$-comodule are equivalent. In general, the property of being an $\mathcal{H}^*$-comodule is stronger than of being an $\mathcal{H}$-module.

Similarly to right $\mathcal{H}^*$-comodule algebras, one can consider left ones. They are also right $\mathcal{H}$-module algebras.

4. FRT- and RE-type algebras

The purpose of this section is to establish a twist-equivalence between certain classes of algebras relative to a quasitriangular Hopf algebra $\mathcal{H}$. Let V be a right $\mathcal{H}$-module of a finite rank over $\mathbb{K}$ and V^* its dual. We identify the space of endomorphisms $\mathrm{End}(V)$ with $V^* \odot V$ and assume V to be a right $\mathrm{End}(V)$-module. The representation ρ of $\mathcal{H}$ on V is a homomorphism $\mathcal{H} \to \mathrm{End}(V)$. The image $(\rho \odot \rho)(\mathcal{R}) \in \mathrm{End}^{\odot 2}(V)$ of the universal R-matrix is denoted R. Given a basis $\{e_i\}$ in V, the elements $e_j^i \in \mathrm{End}(V)$ stand for the matrix units acting on e_i by the rule $e_l e_j^i = e_j \delta_i^l$, where δ_j^l are the Kronecker symbols. The multiplication in $\mathrm{End}(V)$ is expressed according to $e_j^i e_k^l = \delta_j^l e_k^i$.

The space $\mathrm{End}(V)$ is a bimodule for $\mathcal{H}$ or a right module over $\mathcal{H}_{op} \otimes \mathcal{H}$:

$$(30) \qquad A \lhd (x \otimes y) = \rho(x) A \rho(y), \quad A \in \mathrm{End}(V), \quad x \otimes y \in \mathcal{H}_{op} \otimes \mathcal{H}.$$

By duality, the space $\mathrm{End}^*(V)$ is endowed with the structure a left $\mathcal{H}_{op} \otimes \mathcal{H}$-module as well.

4.1. FRT ALGEBRA. Let $\{T_k^i\} \subset \mathrm{End}^*(V)$ be the basis that is dual to $\{e_i^k\}$. The associative algebra $\mathcal{T}_R$ is generated by the matrix coefficients $\{T_k^i\} \subset \mathrm{End}^*(V)$ subject to the FRT relations

$$(31) \qquad\qquad\qquad\qquad R T_1 T_2 = T_2 T_1 R,$$

where T is the matrix $T = \sum_{i,j} T_j^i e_i^j$. The matrix elements T_j^i may be thought of as linear functions on $\mathcal{H}$; they define an algebra homomorphism

$$(32) \qquad\qquad\qquad\qquad \mathcal{T}_R \to \mathcal{H}^*.$$

PROPOSITION 4.1: *Let ρ be a finite dimensional representation of $\mathcal{H}$ and $\mathcal{T}_R$ the FRT algebra associated with ρ. Then $\mathcal{T}_R$ is a $\mathcal{H}$-bimodule algebra, with the left and right actions*

$$(33) \qquad\qquad x \rhd T = T \rho(x), \quad T \lhd x = \rho(x) T, \quad x \in \mathcal{H}$$

extended from $\mathrm{End}^*(V)$. It is a bialgebra, with the coproduct and counit being defined as

$$(34) \qquad\qquad \Delta(T_j^i) = \sum_{l=1}^{n} T_j^l \otimes T_l^i, \quad \varepsilon(T_j^i) = \delta_j^i.$$

Composition of the coproduct with the algebra homomorphism (32) applied to the (left) right tensor factor makes $\mathcal{T}_R$ a (left) right $\mathcal{H}^*$-comodule algebra.

Proof: Actions (33) are extended to the actions on the tensor algebra $\mathrm{T}(\mathrm{End}^*(V))$ leaving invariant the ideal generated by (31). Regarding the bialgebra properties of $\mathcal{T}_R$, the reader is referred for the proof to [FRT]. The comodule structure is inherited from the bialgebra one, so it is obviously coassociative. It is also an algebra homomorphism, being a composition of two homomorphisms. ∎

We remark that the FRT relations (31) arose within the quantum inverse scattering method and was used for systematic definition of the quantum group duals in [FRT].

4.2. RE ALGEBRA.   Another algebra of interest, $\mathcal{L}_R$, is defined as the quotient of the tensor algebra $T(\text{End}^*(V))$ by the RE relations[3]

$$(35) \qquad\qquad R_{21}L_1RL_2 = L_2R_{21}L_1R,$$

where L is the matrix $L = \sum_{i,j} L^i_j e^j_i$ whose entries form the set of generators. In terms of the operator $S = PR$, where $P = \sum_{i,j} e^i_j \odot e^j_i$ is the permutation on $V \odot V$, relations (35) can be written as

$$(36) \qquad\qquad SL_2SL_2 = L_2SL_2S.$$

PROPOSITION 4.2: *Let ρ be a finite dimensional representation of $\mathcal{H}$ and $T^k_l \in \mathcal{H}^*$ its matrix coefficients. Let L^i_j be the generators of the algebra $\mathcal{L}_R$ associated with ρ. Then, $\mathcal{L}_R$ is a left $\mathcal{H}$-module algebra with the action extended from the coadjoint representation in $\text{End}^*(V)$:*

$$(37) \qquad\qquad x \triangleright L = \rho(\gamma(x_{(1)}))L\rho(x_{(2)}), \quad x \in \mathcal{H}.$$

It is a right $\mathcal{H}^*$-comodule algebra with respect to the coaction

$$(38) \qquad\qquad \delta(L^i_j) = \sum_{l,k} L^l_k \odot \gamma(T^k_j)T^i_l.$$

Proof: Action (37) is naturally extended to $T(\text{End}^*(V))$ and preserves relations (36). The coassociativity of (38) is obvious. To prove that δ is an algebra homomorphism, one needs to employ commutation relations (31) and (35). For details, the reader is referred to [KS].   ∎

A spectral dependent version of the RE appeared first in [Cher]. In the form of (35), it may be found in articles [Skl, AFS] devoted to integrable models. The algebra $\mathcal{L}_R$ was studied in [KSk1, KS]. Its relation to the braid group of a solid handlebody was pointed out in [K].

4.3. FRT- AND RE-TYPE ALGEBRAS: TWIST-EQUIVALENCE.

Definition 4.3:   Let $\mathcal{H}$ be a quasitriangular Hopf algebra with the universal R-matrix $\mathcal{R}$ and $\mathcal{A}$ its left module algebra. $\mathcal{A}$ is called **quasi-commutative** if for any $a, b \in \mathcal{A}$

$$(39) \qquad\qquad (\mathcal{R}_2 \triangleright b)(\mathcal{R}_1 \triangleright a) = ab.$$

---

3 Although $\mathcal{T}_R$ and $\mathcal{L}_R$ are both generated by $\text{End}^*(V)$, it is customary to use different letters, T and L, to denote their matrices of generators.

Definition 4.4: We call a quasi-commutative $\mathcal{H}^{op} \odot \mathcal{H}$-module algebra an algebra of **FRT-type**. Similarly, a quasi-commutative $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$-module algebra is called an algebra of **RE-type**.

Example 4.5: Let $\mathcal{A}$ an $\mathcal{H}$-bimodule algebra for a quasitriangular Hopf algebra $\mathcal{H}$ with the universal R-matrix $\mathcal{R}$. As in Example 3.3, we can think of it as a left $\mathcal{H}^{op} \odot \mathcal{H}$-module algebra. Let us take $\mathcal{R}_{13}^{-1} \mathcal{R}_{24}$ for the universal R-matrix of $\mathcal{H}^{op} \odot \mathcal{H}$. The algebra $\mathcal{A}$ is of FRT-type if and only if for any $a, b \in \mathcal{A}$

$$(40) \qquad\qquad (a \triangleleft \mathcal{R}_1)(b \triangleleft \mathcal{R}_2) = (\mathcal{R}_2 \triangleright b)(\mathcal{R}_1 \triangleright a).$$

Example 4.6: The Hopf dual $\mathcal{H}^*$ viewed as an $\mathcal{H}$-bimodule with respect to the coregular actions is an FRT-type algebra. Relation (40) is a consequence of (2).

Example 4.7: The FRT algebra $\mathcal{T}_R$ associated with a finite dimensional representation of $\mathcal{H}$ is of FRT-type. Indeed, it is enough to check commutation relations (4.3) on generators. Reduced to the matrix elements T_j^i generating $\mathcal{T}_R$, condition (40) turns into (31).

PROPOSITION 4.8: *Let $\mathcal{H}$ be a quasitriangular Hopf algebra and let $\tilde{\mathcal{H}} \overset{\mathcal{F}}{\sim} \mathcal{H}$. If $\mathcal{A}$ is a quasi-commutative $\mathcal{H}$-module algebra, then the twisted $\tilde{\mathcal{H}}$-module algebra $\tilde{\mathcal{A}} \overset{\mathcal{F}}{\sim} \mathcal{A}$ is also quasi-commutative.*

Proof: Condition (4.3) holds for twisted multiplication (25) in $\tilde{\mathcal{A}}$ and R-matrix (12) of $\tilde{\mathcal{H}}$. ∎

THEOREM 4.9: *An $\mathcal{H}^{op} \otimes \mathcal{H}$-module algebra is twist-equivalent to an $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$-module algebra.*

Proof: Let $\mathcal{R}_{13}^{-1} \mathcal{R}_{24}$ be the universal R-matrix for $\mathcal{H}^{op} \odot \mathcal{H}$. The twist from $\mathcal{H}^{op} \odot \mathcal{H}$ to $\mathcal{H} \odot \mathcal{H}$ with the twisting cocycle $\mathcal{R}_{13} \in (\mathcal{H}^{op} \otimes \mathcal{H}) \odot (\mathcal{H}^{op} \otimes \mathcal{H})$ turns it into the R-matrix $\mathcal{R}_{31}^{-1} \mathcal{R}_{24}$. Further twist with the cocycle $\mathcal{R}_{23} \in (\mathcal{H} \otimes \mathcal{H}) \odot (\mathcal{H} \odot \mathcal{H})$ transforms $\mathcal{H} \otimes \mathcal{H}$ into $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$ with the R-matrix $\mathcal{R}_{41}^{-1} \mathcal{R}_{31}^{-1} \mathcal{R}_{24} \mathcal{R}_{23}$, see (14). ∎

COROLLARY 4.10: *Let $\mathcal{A}$ be an $\mathcal{H}$-bimodule algebra and $\tilde{\mathcal{A}}$ its twist-equivalent left $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$-module algebra. The algebra $\tilde{\mathcal{A}}$ is of RE-type if and only if $\mathcal{A}$ is of FRT-type. Then, for any $a, b \in \tilde{\mathcal{A}}$*

$$(41) \qquad (\mathcal{R}_{1'} \triangleright a \triangleleft \mathcal{R}_2)(b \triangleleft \mathcal{R}_1 \triangleleft \mathcal{R}_{2'}) = (\mathcal{R}_{1'} \triangleright \mathcal{R}_2 \triangleright b)(\mathcal{R}_1 \triangleright a \triangleleft \mathcal{R}_{2'}),$$

where the primes distinguish different copies of $\mathcal{R}$.

Proof: By Proposition 4.8, $\tilde{\mathcal{A}}$ is of RE-type *iff* $\mathcal{A}$ is of FRT-type; then $\mathcal{A}$ satisfy (40). Twist from $\mathcal{H}^{op} \otimes \mathcal{H}$ to $\mathcal{H} \overset{R}{\otimes} \mathcal{H}$ converts (40) to (41). The action of $\mathcal{H} \overset{R}{\otimes} \mathcal{H}$ on $\tilde{\mathcal{A}}$ is expressed through the right and left actions of $\mathcal{H}$ on $\mathcal{A}$ as in (24). ∎

PROPOSITION 4.11: *Let ρ be a finite dimensional representation of $\mathcal{H}$. The RE algebra $\mathcal{L}_R$ associated with ρ is an RE-type algebra, provided the universal R-matrix is taken as in (14). It is twist-equivalent to $\mathcal{T}_R$.*

Proof: First, let us prove that $\mathcal{L}_R$ is an $\mathcal{H} \overset{R}{\otimes} \mathcal{H}$-module algebra. Define the left $\mathcal{H} \overset{R}{\otimes} \mathcal{H}$-action on $\mathrm{End}^*(V)$ by the formula

$$(42) \qquad (x \otimes y) \rhd L = \rho(\gamma(x))L\rho(y), \quad x \otimes y \in \mathcal{H} \overset{R}{\otimes} \mathcal{H},$$

where $\{L_j^i\} \subset \mathrm{End}^*(V)$ is the dual basis to $\{e_i^j\}$. This action is uniquely extended to $\mathrm{T}(\mathrm{End}^*(V))$ to make it an $\mathcal{H} \overset{R}{\otimes} \mathcal{H}$-module algebra. It is easy to see that action (42) respects relations (35) thus reducing to an action on $\mathcal{L}_R$. Now we show that $\mathcal{L}_R$ is of RE-type. Abstract commutation relations (4.3) are specialized to (41) for quasi-commutative $\mathcal{H} \overset{R}{\otimes} \mathcal{H}$-module algebras. Evaluated on the generators L_j^i, they turn into the reflection equation relations (35). On the other hand, as mentioned in Example 4.7, the FRT relations (31) are the reduction of (40) to the generators of $\mathcal{T}_R$. The twist that relates identities (40) and (41) transforms relations (31) to (35). ∎

4.4. RE DUAL $\tilde{\mathcal{H}}^*$ AND ITS PROPERTIES. Theorem 4.9 allows us to define the RE analog of the algebra $\mathcal{H}^*$ for any quasitriangular Hopf algebra.

Definition 4.12: **Reflection equation dual $\tilde{\mathcal{H}}^*$** to a quasitriangular Hopf algebra $\mathcal{H}$ is an RE-type algebra, the twist of $\mathcal{H}^*$ viewed as a coregular left $\mathcal{H}^{op} \otimes \mathcal{H}$-module with action (24).

Let m be the multiplication in $\mathcal{H}^*$. Formula (25) gives the multiplication in $\tilde{\mathcal{H}}^*$ which is expressed through the universal R-matrix and the coregular actions of $\mathcal{H}$ on $\mathcal{H}^*$:

$$(43) \qquad \tilde{m}(a \otimes b) = m(\mathcal{R}_1 \rhd a \lhd \mathcal{R}_{1'} \otimes b \lhd \gamma(\mathcal{R}_2) \lhd \mathcal{R}_{2'}).$$

It follows that $\tilde{\mathcal{H}}^*$ is isomorphic as an associative algebra to the braided Hopf algebra introduced by Majid using other arguments, [Mj].

PROPOSITION 4.13: $\tilde{\mathcal{H}}^*$ is a left module algebra over the Hopf algebras $\mathcal{H}$, $\mathcal{H}^{*op}$, and the double $D(\mathcal{H})$.

Proof: An immediate corollary of Proposition 2.5 and Theorem 2.6. ∎

Let $(\mathcal{H}\overset{R}{\otimes}\mathcal{H})^*$ be the dual Hopf algebra to $\mathcal{H}\overset{R}{\otimes}\mathcal{H}$. As a coalgebra, it coincides with the tensor product $\mathcal{H}^* \odot \mathcal{H}^*$. The multiplication in $(\mathcal{H}\overset{R}{\otimes}\mathcal{H})^*$ is expressed through coregular actions (22) of the universal R-matrix of $\mathcal{H}$:

$$(\alpha \odot \beta)(\eta \otimes \xi) = \alpha(\mathcal{R}_2 \rhd \eta \lhd \mathcal{R}_{2'}^{-1}) \odot (\mathcal{R}_1 \rhd \beta \lhd \mathcal{R}_{1'}^{-1})\xi,$$

for $\alpha \otimes \beta$ and $\eta \otimes \xi \in (\mathcal{H}\overset{R}{\otimes}\mathcal{H})^*$.

PROPOSITION 4.14: $\tilde{\mathcal{H}}^*$ is a right $(\mathcal{H}\overset{R}{\otimes}\mathcal{H})^*$-comodule algebra with the coaction expressed through the coproduct and antipode in $\mathcal{H}^*$:

$$(44) \qquad \delta(\eta) = \eta_{(2)} \odot (\gamma(\eta_{(1)}) \odot \eta_{(3)}), \quad \eta \in \tilde{\mathcal{H}}^*.$$

Proof: Map (44) satisfies condition (29) with left action (24). ∎

COROLLARY 4.15: $\tilde{\mathcal{H}}^*$ is a right $\mathcal{H}^*$-comodule algebra with the coaction

$$(45) \qquad \delta(\eta) = \eta_{(2)} \odot \gamma(\eta_{(1)})\eta_{(3)}, \quad \eta \in \tilde{\mathcal{H}}^*.$$

$\tilde{\mathcal{H}}^*$ is a right $\mathcal{H}_{op}$-comodule algebra, with the coaction

$$(46) \qquad \delta(\eta) = \eta_{(2)} \odot \langle \eta_{(3)}, \mathcal{R}_{1'}^{-1}\rangle\langle\gamma(\eta_{(1)}), \mathcal{R}_2\rangle\mathcal{R}_{2'}^{-1}\mathcal{R}_1, \quad \eta \in \tilde{\mathcal{H}}^*,$$

where $\langle ., .\rangle$ is the Hopf pairing between $\mathcal{H}^*$ and $\mathcal{H}$.

Proof: The Hopf embedding (16) gives rise to the reversed arrow $(\mathcal{H}\overset{R}{\otimes}\mathcal{H})^* \to \mathcal{H}^*$ acting by multiplying the tensor factors. It remains to apply this homomorphism to the term in (44) that is confined in parentheses, to obtain (45). Similarly, evaluating the map $(\mathcal{H}\overset{R}{\otimes}\mathcal{H})^* \to \mathcal{H}_{op}$, which is dual to the Hopf algebra homomorphism (17), we come to (46). ∎

LEMMA 4.16: The counit ε of $\mathcal{H}^*$ is a character of the RE dual $\tilde{\mathcal{H}}^*$

Proof: Applying ε to the twisted product (43) we find, for $a, b \in \tilde{\mathcal{H}}^*$,

$$\varepsilon \circ \tilde{m}(a \odot b) = \varepsilon \circ m(\mathcal{R}_1 \rhd a \lhd \mathcal{R}_{1'} \otimes b \lhd \gamma(\mathcal{R}_2) \lhd \mathcal{R}_{2'})$$
$$= \langle a, \mathcal{R}_{1'}\mathcal{R}_1\rangle\langle b, \gamma(\mathcal{R}_2)\mathcal{R}_{2'}\rangle = \varepsilon(a)\varepsilon(b).$$

We used the identity $\mathcal{R}_{1'}\mathcal{R}_1 \otimes \gamma(\mathcal{R}_2)\mathcal{R}_{2'} = 1 \odot 1$ following from (4). ∎

PROPOSITION 4.17: *Let $\mathcal{H}$ be a quasitriangular Hopf algebra with the universal R-matrix $\mathcal{R}$. Consider the element $\mathcal{Q} = \mathcal{R}_{21}\mathcal{R} \in \mathcal{H}^{\odot 2}$. The map $\tilde{\mathcal{H}}^* \to \mathcal{H}$ defined by the correspondence*

$$(47) \qquad\qquad \eta \mapsto \langle \eta, \mathcal{Q}_1 \rangle \mathcal{Q}_2, \quad \eta \in \tilde{\mathcal{H}}^*,$$

where $\langle .,. \rangle$ is the Hopf pairing between $\mathcal{H}^*$ and $\mathcal{H}$, is a homomorphism of the coadjoint and adjoint $\mathcal{H}$-module algebras.

Proof: According to Lemma 4.16, the counit of $\mathcal{H}^*$ is a character of $\tilde{\mathcal{H}}^*$. Applying the antipode to the right tensor factor of (46) we pass to a left comodule structure with respect to the Hopf algebra $\mathcal{H}^{op}$. Applying the counit to the left tensor factor belonging to $\tilde{\mathcal{H}}^*$, we obtain an equivariant homomorphism of $\mathcal{H}$-module algebras $\tilde{\mathcal{H}}^* \to \mathcal{H}$. Taking composition of $(\varepsilon \odot \gamma)$ with coaction (46) we obtain (47):

$$(48) \qquad \begin{aligned} (\varepsilon \odot \gamma) \circ \delta(\eta) &= \langle \eta_{(2)}, \mathcal{R}_{1'}^{-1} \rangle \langle \gamma(\eta_{(1)}), \mathcal{R}_2 \rangle \gamma(\mathcal{R}_{2'}^{-1}\mathcal{R}_1) \\ &= \langle \eta, \gamma(\mathcal{R}_2)\mathcal{R}_{1'}^{-1} \rangle \gamma(\mathcal{R}_1)\gamma(\mathcal{R}_{2'}^{-1}) = \langle \eta, \mathcal{R}_2\mathcal{R}_{1'} \rangle \mathcal{R}_1\mathcal{R}_{2'}. \end{aligned}$$

Here we used $(\gamma \odot \gamma)(\mathcal{R}) = \mathcal{R}$ and $(\text{id} \odot \gamma)(\mathcal{R}^{-1}) = \mathcal{R}$, see (4). Equivariance of map (47) may be derived from the comodule structure (46). However, it is readily seen directly. Since $\Delta(x)\mathcal{Q} = \mathcal{Q}\Delta(x)$ for every $x \in \mathcal{H}$, one has $\gamma(x_{(1)})\mathcal{Q}_1 x_{(2)} \odot \mathcal{Q}_2 = \mathcal{Q}_1 \odot x_{(1)}\mathcal{Q}_2\gamma(x_{(2)})$. Then,

$$(49) \quad \langle x_{(2)} \triangleright \eta \triangleleft \gamma(x_{(1)}), \mathcal{Q}_1 \rangle \mathcal{Q}_2 = \langle \eta, \gamma(x_{(1)})\mathcal{Q}_1 x_{(2)} \rangle \mathcal{Q}_2 = \langle \eta, \mathcal{Q}_1 \rangle x_{(1)}\mathcal{Q}_2\gamma(x_{(2)}),$$

for $x \in \mathcal{H}$ and $\eta \in \tilde{\mathcal{H}}^*$.   ∎

5. Applications: equivariant quantization on G-spaces

In this section, we specialize the constructions of the previous sections to the deformation quantization situation when $\mathcal{H} = \mathcal{U}_h(\mathfrak{g})$, a quantum group corresponding to a semisimple Lie algebra $\mathfrak{g}$.

5.1. QUANTIZATION ON THE GROUP SPACE G.   Let G be a semisimple Lie group equipped and $\mathfrak{g}$ its Lie algebra. An element $\xi \in \mathfrak{g}$ generates left and right invariant vector fields

$$(\xi^l \triangleright f)(g) = \frac{d}{dt}f(ge^{t\xi})|_{t=0}, \quad (f \triangleleft \xi^r)(g) = \frac{d}{dt}f(e^{t\xi}g)|_{t=0},$$

defining the left and right actions of the algebra $\mathcal{U}(\mathfrak{g})$ on functions on G. Given an element $\psi \in \mathcal{U}(\mathfrak{g})$, by ψ^r and ψ^l we correspondingly denote its extensions

by the right- and left-invariant differential operators on G. We use notation $\xi^{\mathrm{ad}} = \xi^l - \xi^r$ for the vector field generated by the element $\xi \in \mathfrak{g}$ via the adjoint action $a \mapsto g^{-1}ag$ of G on itself.

Let $r \in \wedge^2\mathfrak{g}$ be a classical r-matrix and ω the invariant symmetric element such that $r + \omega$ satisfies the classical Yang–Baxter equation, [Dr1]. Let $\mathcal{U}_h(\mathfrak{g})$ be the corresponding quantum group.

PROPOSITION 5.1: *The RE dual $\tilde{\mathcal{U}}_h^*(\mathfrak{g})$ to the quantum group $\mathcal{U}_h(\mathfrak{g})$ is a $\mathcal{U}_h(\mathfrak{g})\overset{\mathcal{R}}{\bar{\otimes}}\mathcal{U}_h(\mathfrak{g})$-equivariant quantization of the Poisson bracket*

$$(50) \qquad r^{r,r} + r^{l,l} - r^{l,r} - r^{r,l} + (\omega^{r,l} - \omega^{l,r})$$

on the group G. It turns into a $\mathcal{U}_h(\mathfrak{g})$-equivariant quantization of the bracket

$$(51) \qquad r^{\mathrm{ad},\mathrm{ad}} + (\omega^{r,l} - \omega^{l,r})$$

via the Hopf algebra embedding $\Delta\colon \mathcal{U}_h(\mathfrak{g}) \to \mathcal{U}_h(\mathfrak{g})\overset{\mathcal{R}}{\bar{\otimes}}\mathcal{U}_h(\mathfrak{g})$.

Proof: The element $r^- = (-r) \oplus r$ from the exterior square $\wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ generates the bivector field $r_G^- = -r^{r,r} + r^{l,l}$ on G via the action of $\mathcal{U}(\mathfrak{g})_{op} \otimes \mathcal{U}(\mathfrak{g})$; r_G^- coincides with the Drinfeld–Sklyanin bracket on G. The algebra $\mathcal{U}_h^*(\mathfrak{g})$ is the $\mathcal{U}_h(\mathfrak{g})^{op} \otimes \mathcal{U}_h(\mathfrak{g})$-equivariant quantization of r_G^-, [Ta]. The twist with the cocycle $\mathcal{R}_{13} = 1 + h(r_{13} + \omega_{13}) + o(h)$ converts $\mathcal{U}_h^*(\mathfrak{g})$ into the $\mathcal{U}_h(\mathfrak{g}) \otimes \mathcal{U}_h(\mathfrak{g})$-equivariant quantization of the bracket $r^{r,r} + r^{l,l}$. Indeed, at the infinitesimal level this procedure adds the term $2r^{r,r}$ to the bracket r_G^-. Further twist with the cocycle $\mathcal{R}_{23} = 1 + h(r_{23} + \omega_{23}) + o(h)$ leads to the algebra $\tilde{\mathcal{U}}_h^*(\mathfrak{g})$. In terms of Poisson brackets, this operation adds the term $-(r^{l,r} + \omega^{l,r}) - (r^{r,l} - \omega^{r,l})$ to $r^{r,r} + r^{l,l}$ thus resulting in (50). The last statement of the proposition is straightforward. ∎

5.2. QUANTIZATION OF POLYNOMIAL FUNCTIONS ON MATRICES. Let V be a complex vector space and ρ a homomorphism of $\mathcal{U}(\mathfrak{g})$ into the matrix algebra $\mathcal{M} = \mathrm{End}(V)$. It induces a bimodule structure on $\mathcal{M}^*$:

$$(x \triangleright f)(A) = f(A\rho(x)), \quad (f \triangleleft x)(A) = f(\rho(x)A),$$

for $x \in \mathcal{U}(\mathfrak{g})$, $f \in \mathcal{M}^*$, and $A \in \mathcal{M}$. Let $\Omega \in \mathcal{M}^{\otimes 2}$ be the image of the invariant symmetric element ω. Denote by $\mathcal{M}^\Omega$ the cone of matrices

$$\mathcal{M}^\Omega = \{A \in \mathcal{M} \mid [\Omega, A \otimes A] = 0\}.$$

Evidently, $\mathcal{M}^\Omega$ is an algebraic variety, it is closed under the matrix multiplication and invariant with respect to the two-sided action of the group G.

Remark 5.2: We would like to stress that we do not restrict the consideration to fundamental representations of $\mathcal{U}(\mathfrak{g})$. The subspace $\mathcal{M}^\Omega$ coincides with $\mathcal{M} =$ End(V) only for $\mathfrak{g} = sl(n, \mathbb{C})$ and $V = \mathbb{C}^n$.

PROPOSITION 5.3: *The quotient of the algebra $\mathcal{L}_R$ by the torsion is a $\mathcal{U}_h(\mathfrak{g}) \overset{\mathcal{R}}{\otimes} \mathcal{U}_h(\mathfrak{g})$-equivariant quantization of the Poisson bracket*

$$(52) \qquad r^{r,r} + r^{l,l} - r^{l,r} - r^{r,l} + (\omega^{r,l} - \omega^{l,r}).$$

Proof: It is easy to see that $\mathcal{M}^\Omega$ is the maximal subspace in $\mathcal{M}$ where bracket (52) is Poisson. It was proven in [DS] that the algebra $\mathcal{T}_R$ is a $\mathcal{U}_h(\mathfrak{g})_{op} \otimes \mathcal{U}_h(\mathfrak{g})$-equivariant quantization on $\mathcal{M}^\Omega$. Applying the RE twist, we obtain the algebra $\mathcal{L}_R$ as the quantization on $\mathcal{M}^\Omega$. This twist transforms the bracket $-r^{l,l} + r^{r,r}$ on $\mathcal{M}^\Omega$ to bracket (52). This proves the statement.   ∎

Note that bracket (52) goes over into $r^{\mathrm{ad},\mathrm{ad}} + (\omega^{r,l} - \omega^{l,r})$ after restriction of $\mathfrak{g} \oplus \mathfrak{g}$ to the diagonal subalgebra. Then $\mathcal{L}_R$ becomes an equivariant quantization of this bracket with respect to $\mathcal{U}_h(\mathfrak{g}) \subset \mathcal{U}_h(\mathfrak{g}) \overset{\mathcal{R}}{\otimes} \mathcal{U}_h(\mathfrak{g})$.

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